System identification and metrology for quantum Markov processes

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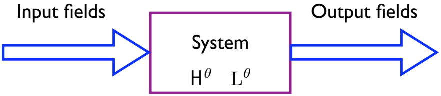
EPSRC

Quantum Control Enginering: Mathematical Principles and Applications

Isaac Newton Institute, Cambridge, July 2014

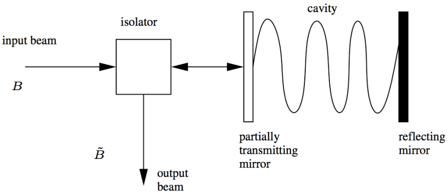
The University of Nottingham

Open system in the input-output formalism

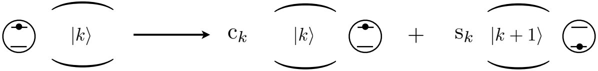


- system interacts with environment in a Markov fashion
- Unitary dynamics: quantum stochastic differential equations [1] [2]
- Goal: comprehensive theory of the output process and applications
  - Central Limit Theorem for fluctuations $\longrightarrow$ statistical theory of system identification
  - Large Deviations $\longrightarrow$ thermodynamics of quantum trajectories[3]
  - Ergodicity of quantum trajectories $\longrightarrow$ filtering theory (stability, purification)
  - Applications in quantum metrology and quantum control

[1] K. R. Parthasarathy, *An introduction to quantum stochastic calculus*, Birkhäuser 1992
[2] C. W. Gardiner and P. Zoller, *Quantum Noise*, Springer 2004
[3] J.P. Garrahan and I. Lesanovsky, *Phys. Rev. Lett.* 2010



Linear system $\longrightarrow$ see presentation by N. Yamamoto[4]



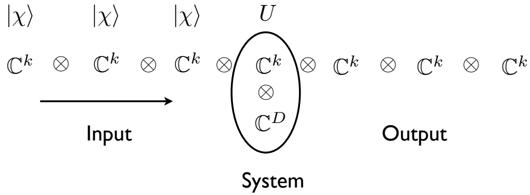
Non-linear system: atom maser $\longrightarrow$ this presentation

[4] M.G. and N. Yamamoto, Proc. IEEE CDC 2013

Outline

- Discrete time quantum Markov chains
  - output state, ergodicity
  - Central Limit Theorem for time averages
- Identifiability
  - which dynamical parameters can be identified ?
- Estimation precision
  - what is the quantum/classical Fisher information ?
  - are the errors normally distributed ?
- Quantum enhanced metrology
  - which Markov systems exhibit Heisenberg scaling ?
  - Is the Markov setting suitable for generating metrologically useful states ?

Successive unitary interactions of "memory" with "quantum noise units"



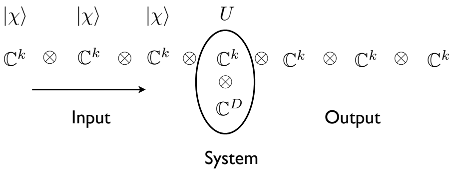
- output state is closely related to finitely correlated[5] and matrix product states[6]
- quantum analogue of the ‘classical’ hidden Markov chains
- standard tools[7] for switching between discrete and continuous time

[5] M. Fannes, B. Nachtergale and R. Werner, Commun. Math. Phys. 1992

[6] D. Perez-Garcia, F. Verstraete, M. Wolf and I. Cirac, Quantum Inf. Comput. 2007

[7] thanks to R. Hudson's work and others

Output state



- Dynamics depends only on the isometry

$$
\begin{aligned}
V : \mathbb{C}^D &\to \mathbb{C}^D \otimes \mathbb{C}^k \\
V : |\varphi\rangle &\mapsto U|\varphi\rangle \otimes |\chi\rangle = \sum_i K_i|\varphi\rangle \otimes |i\rangle
\end{aligned}
$$

- System + output state as MPS

$$
|\Psi_{V,\varphi}(n)\rangle := V^{(n)} \cdots V^{(1)}|\varphi\rangle = \sum_{(i_1,\dots,i_n)} K_{i_n} \cdots K_{i_1}|\varphi\rangle \otimes |i_n \otimes \cdots \otimes i_1\rangle
$$

- System's transition operator

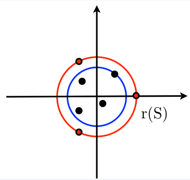
$$
T_V(\rho) = \sum_i K_i \rho K_i^*
$$

Primitive (mixing) chains and quantum Perron-Frobenius Theorem [8]

Quantum Perron-Frobenius Theorem

- If S is an irreducible CP map (no invariant subspace)
  - The spectral radius $r(S)$ is a non-degenerate eigenvalue of S
  - the corresponding eigenvector is a strictly positive matrix
  - the eigenvalues on the unit circle form a group
- If T is a transition operator (CP & TP) then
  - $r(T) = 1$
  - unique full rank stationary state: $T(\rho_{ss}) = \rho_{ss}$
- If T is primitive (irreducible and aperiodic) then
  - $|\lambda| < 1$ for all remaining eigenvalues
  - convergence to stationary state

$$T^n(\sigma) = (|\rho_{ss}\rangle\langle\mathbf{1}| + \text{Rest})^n\,|\sigma\rangle = (|\rho_{ss}\rangle\langle\mathbf{1}| + \text{Rest}^n)\,|\sigma\rangle \longrightarrow \rho_{ss}$$

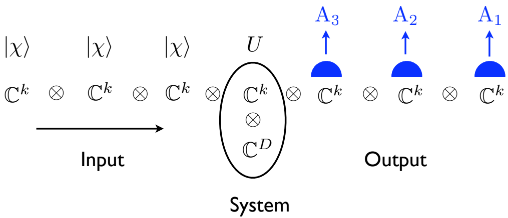


Key observation: if T_ϵ is a small perturbation of primitive $T \Longrightarrow$ dominant eigenvalue λ_ϵ is smooth and determines the asymptotics

$$\text{“ } T_\epsilon^n \approx \lambda_\epsilon^n \text{ ”}$$

[8] D. E. Evans and R. Hoegh-Krohn, *J. London Math. Soc*, 1978; M. Sanz *et al*, *IEEE Trans. Inform. Th.*, 2010

Quantum Markov chains: sequential output measurements



- Observable $A = \sum_i a_i |i\rangle\langle i|$ is measured on each unit $\longrightarrow$ outcomes $A_1, A_2, \ldots$
- Basic statistic: time (empirical) average $S_n(A) := \frac{1}{n}\sum_{i=1}^n A_i$
- Moment generating function

$$\phi(s) := \mathbb{E}\left(e^{snS_n(A)}\right) = \mathrm{Tr}\left(T_s^n(\rho_{in})\right)$$

- Deformed transition operator $T_s : M(\mathbb{C}^D) \to M(\mathbb{C}^D)$ (CP, non-TP):

$$T_s : \rho \longmapsto \sum_i e^{sa_i} K_i \rho K_i^*$$

Central Limit Theorem for the measurement process[11]

Theorem (Central Limit)

Let T be primitive. Then
1) time averages converge to stationary means: $S_n(A) \longrightarrow \mathbb{E}_{ss}(A)$

2) fluctuations are normal

$$\mathbb{F}_n(A) := \sqrt{n}\,(S_n(A) - \mathbb{E}_{ss}(A)) \xrightarrow[n\to\infty]{\mathcal{L}} N(0, V(A))$$

with variance

$$V(A) = \begin{cases} \mathbb{E}_{ss}\left(A^2\right) + 2\mathbb{E}_{ss}\left(A \otimes (\mathrm{Id} - T)^{-1}(B)\right), & B = \langle\psi|U^*(\mathbf{1} \otimes A)U|\psi\rangle \\ \left.\frac{d^2 \log \lambda_s}{ds^2}\right|_{s=0}, & \lambda_s = \text{dominant eigenvalue of } T_s. \end{cases}$$

Remarks

1) A similar CLT holds for time averages of multiple-outcomes functions $f(A_1, \ldots, A_r)$, and it involves an extended deformed transition operator T_f which keeps track of previous outcomes [9]

2) A similar CLT holds for the total counts and integrated homodyne current in continuous-time measurements [10]

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[9] M. van Horssen and M.G., arXiv:1407.5082

[10] C. Catana, L. Bouten and M.G., arXiv:1407.5131

[11] B. Kummerer and H. Maaseen, *J. Phys. A: Math. Gen.* 2003; M.G. Phys. Rev. A, 2011

Ideas of the proof

- CLT follows from the convergence of characteristic functions

$$\mathbb{E}\left(e^{is\mathbb{F}_n(A)}\right) = \mathrm{Tr}\left(T^n_{is/\sqrt{n}}(\rho_{in})\right) \xrightarrow[n\to\infty]{} e^{-s^2V(A)/2}$$

- A) Use non-zero spectral gap and second order expansion

$$T_{is/\sqrt{n}} = T_0 + \frac{1}{\sqrt{n}}T_1 + \frac{1}{n}T_2 + \text{Rest}$$

to show

$$T^n_{is/\sqrt{n}}(\mathbf{1}) = \left(1 - \frac{s^2V(A)}{2n}\right)^n \mathbf{1} + o(1) \to \exp(-s^2V(A)/2)\mathbf{1}$$

- B) Follows (together with large deviations) from

$$\mathrm{Tr}\left(T^n_{is/\sqrt{n}}(\rho_{in})\right) \approx e^{n\log\lambda_{is/\sqrt{n}}} \approx \exp\left(-\frac{s^2}{2}\left.\frac{d^2\log\lambda_s}{ds^2}\right|_{s=0}\right)$$

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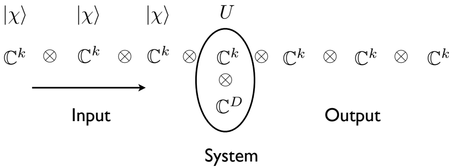
The system identification problem



- Suppose that the dynamics depends on an unknown parameter θ, so $U = U_\theta$
- Task: estimate the parameter by measuring the output state
- Methods: Bayesian/extended filter[12], maximum likelihood, compressed sensing[13]
- Here we are interested in statistical aspects
  - which parameters are identifiable?
  - what is the accuracy / Fisher information?
  - what is the structure of the output state?

[12] H. Mabuchi *Quant. Semiclass. Optics* 1996; J. Gambetta and H. M. Wiseman *Phys. Rev. A* 2001; S. Gammelmark and K. Mølmer *Phys. Rev. A* 2013

[13] M. Cramer *et al*, *Nat. Commun.* 2010

Equivalence classes of Markov chains with identical (stationary) outputs[16]

Definition

Two primitive chains with isometries V_1 and V_2 are called equivalent if for all n,

$$\rho^{out}_{V_1}(n) = \rho^{out}_{V_2}(n).$$

Theorem (equivalence classes)

Two primitive chains with isometries V_1 and V_2 are equivalent if and only if there exists a phase $e^{i\phi}$ and a unitary $W : \mathbb{C}^D \to \mathbb{C}^D$ such that

$$V_2 = e^{i\phi}(W \otimes \mathbf{1})V_1 W^*$$

or equivalently

$$K_i^{V_2} = e^{i\phi} W\, K_i^{V_1}\, W^*, \qquad i = 1, \dots, k.$$

Remarks

1) Theorem 1 is a quantum extension of the 'classical' result by Petrie[14] on equivalence classes of ergodic hidden Markov chains

2) similar result holds in continuos-time: $L_i^{V_2} = W L_i^{V_1} W^*$ and $H^{V_2} = W H^{V_1} W^* + cI$

3) similar result holds for (passive) linear systems [15] [see presentation by N. Yamamoto]

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[14] T. Petrie, *Annals of Math. Statistics*, 1969

[15] M.G. and N. Yamamoto, *IEEE Proceedings 52nd CDC* 2013

[16] M.G. and J. Kiukas, arXiv:1402.3535

Idea of the proof

- Define the "off-diagonal transition operator"

$$T_{12} : \rho \mapsto \sum_{i=1}^{D} K_i^{V_1} \rho K_i^{V_2 *}$$

- Overlap of the two system-output states

$$\langle \Psi_{V_2,\varphi}(n) | \Psi_{V_1,\varphi}(n) \rangle = \mathrm{Tr}\left(T_{1,2}^n (|\varphi\rangle\langle\varphi|) \right) \approx \lambda_{1,2}^n$$

- Two alternatives:

A) $|\lambda_{12}| = 1 \implies K_i^{V_2} = e^{i\phi} W \, K_i^{V_1} \, W^* \implies$ equivalent systems

B) $|\lambda_{12}| < 1 \implies$ overlap decays exponentially $\implies$ non-equivalent systems

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Quantum statistics for pure states: information in overlap

- Statistical problem:

  estimate parameter $\theta \in \mathbb{R}$ by measuring ensemble of n i.i.d. systems with state

$$|\psi_\theta\rangle = e^{i\theta G}|\psi_0\rangle, \qquad \langle\psi_0|G|\psi_0\rangle = 0.$$

- Local asymptotic normality a.k.a. Holstein-Primakov:

  In an "uncertainty neighbourhood" of size $n^{-1/2}$, the overlaps of joint states are approximately Gaussian

$$\left\langle \psi^{\otimes n}_{\theta_0 + v/\sqrt{n}} \middle| \psi^{\otimes n}_{\theta_0 + u/\sqrt{n}} \right\rangle = \left\langle \psi_0 \left| e^{i(u-v)G/\sqrt{n}} \right| \psi_0 \right\rangle^n \longrightarrow e^{\frac{(u-v)^2 F}{8}} = \left\langle \sqrt{F/2}v \middle| \sqrt{F/2}u \right\rangle$$

  - $F = 4\mathrm{Var}(G) = 4\left\|\frac{d\psi_\theta}{d\theta}\right\|^2$ is the quantum Fisher information
  - $\left|\sqrt{F/2}u\right\rangle$ is a one-mode coherent state with displacement $\left(\sqrt{F/2}u, 0\right)$

- Remark

  LAN holds for mixed states &multi-dimensional models, and has an operational interpretation in terms of mutual approximation through quantum channels [17]

[17] J. Kahn and M.G., *Commun. Math. Phys.*, 2009

System identification: system+output setup[19]

- Primitive Markov chain $V = V_\theta$ with $\theta \in \mathbb{R}$ unknown parameter
- 'Localise' θ as $\theta = \theta_0 + \frac{u}{\sqrt{n}}$ by using adaptive measurements
- Assume complete access to system + output state $|\Psi_{u,\varphi}(n)\rangle \equiv |\Psi_{V_\theta,\varphi}(n)\rangle$

Theorem (LAN for quantum Markov chains)

The output model $|\Psi_{u,\varphi}(n)\rangle$ converges 'weakly' to the coherent state model $\left|\sqrt{F/2}\,u\right\rangle$

$$\lim_{n\to\infty} \langle \Psi_{v,\varphi}(n) | \Psi_{u,\varphi}(n)\rangle = \left\langle \sqrt{F/2}\,v \middle| \sqrt{F/2}u \right\rangle = \exp(-F(u-v)^2/8)$$

→ F is the quantum Fisher information 'per sample'

→ optimal estimator satisfies asymptotic normality

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} N(0, F^{-1})$$

Remark: the above Theorem is a quantum extension of 'classical' results on HMC [18]

[18] P.J. Bickel, Y. Ritov, and T. Ryden *Ann. Statist* (1998)

[19] M.G. and J. Kiukas, arXiv:1402.3535

Quantum Fisher information as susceptibility

- Overlap can be reduced to dominant eigenvalue of a deformed transition operator

$$\langle \Psi_{v,\varphi}(n) | \Psi_{u,\varphi}(n) \rangle = \mathrm{Tr}\left(T^n_{\frac{u}{\sqrt{n}},\frac{v}{\sqrt{n}}} (|\varphi\rangle\langle\varphi|) \right) \approx \exp(n \log \lambda_{\frac{u}{\sqrt{n}},\frac{v}{\sqrt{n}}})$$

- Expanding in $\frac{u}{\sqrt{n}}, \frac{v}{\sqrt{n}}$, and setting $\frac{\partial \lambda a,b}{\partial a}\Big|_{a=b=0} = 0$

$$\langle \Psi_{v,\varphi}(n) | \Psi_{u,\varphi}(n) \rangle \longrightarrow \exp\left(\frac{1}{2} \left. \frac{\partial^2 \log \lambda_{a,b}}{\partial a \partial b} \right|_{a=b=0} \right)$$

so that

$$F = -4 \left. \frac{\partial^2 \log \lambda_{a,b}}{\partial a \partial b} \right|_{a=b=0}$$

- Similar methods have been used in[20], [21]

[20] M.Cozzini, R. Ionicioiu, and P. Zanardi, *Phys. Rev. B*, 2007

[21] S. Gammelmark and K. Mølmer, *Phys. Rev. Lett.*, 2014

Quantum Fisher information as variance of a "Markov generator"

Quantum Fisher information = Markovian variance of the generator $G := i\dot{U}U^*$

$$F = 4\mathrm{Var}_V(G) = 4(G,G)_V$$

Fluctuations operator: for $X \in M_D \otimes M_k$ with $\mathbb{E}_{ss}(X) = 0$

$$\mathbb{F}_n(X) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X(i), \qquad X(i) = U(i)^* X_i U(i)$$

Markov covariance inner product: for $X, Y \in M_D \otimes M_k$

$$\begin{aligned} (X,Y)_V &:= \lim_{n\to\infty} \frac{1}{n} \langle \mathbb{F}_n(X^*)\mathbb{F}_n(Y)\rangle \\ &= \mathbb{E}_{ss}\left[X^*Y + X^*\left(\mathcal{R}\circ\mathcal{E}(Y)\otimes \mathbf{1}\right) + \left(\mathcal{R}\circ\mathcal{E}(X^*)\otimes\mathbf{1}\right)Y\right] \end{aligned}$$

where

- $\mathcal{E} : M_D \otimes M_k \to M_D$ is the conditional expectation $\mathcal{E}(X) = V^* X V$
- $\mathcal{R} = (\mathrm{Id} - T)^{-1}\,|_{\mathcal{D}}$ with $\mathcal{D} := \{Y : \mathrm{Tr}(\rho_{ss} Y) = 0\}$

System identification: output alone setup[22]

- Assume that dynamics is in the stationary regime
- Assume access to output state $\rho_u^{out}(n) := \rho_{V_\theta}^{out}(n)$

Theorem (LAN for quantum Markov chains)

The output model $\rho_u^{out}(n)$ converges strongly to the coherent state model $\left|\sqrt{F/2}\,u\right\rangle$: there exists quantum channels T_n and S_n such that

$$\lim_{n\to\infty} \sup_{|u|<C} \left\| T_n(\rho_u^{out}(n)) - \left|\sqrt{F/2}\,u\right\rangle\left\langle\sqrt{F/2}\,u\right| \right\|_1 = 0$$

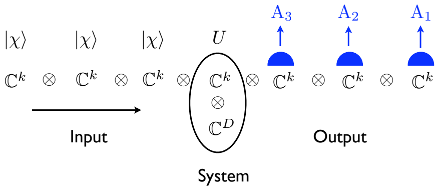
$$\lim_{n\to\infty} \sup_{|u|<C} \left\| \rho_u^{out}(n) - S_n\left(|\sqrt{F/2}\,u\rangle\langle\sqrt{F/2}\,u|\right) \right\|_1 = 0$$

→ F is the quantum Fisher information 'per sample'

→ "output contains all information"

[22] M.G. and J. Kiukas, arXiv:1402.3535

Classical Fisher information of time averages



- Unknown dynamics V_θ with parameter $\theta = \theta_0 + u/\sqrt{n}$

- Time average $S_n = \frac{1}{n}\sum_{i=1}^{n} A_i$ captures deviations from mean $\mu_{\theta_0} = \mathbb{E}_{\theta_0}(A)$

$$\sqrt{n}(S_n - \mu_{\theta_0}) \xrightarrow[n\to\infty]{\mathcal{L}} N\left(\frac{d\mu}{d\theta}u \,, V(A)\right)$$

- Classical Fisher information = signal to noise ratio (in terms of dom. eigenv. $\lambda_{s,\theta}$ of $T_{s,\theta}$)

$$I^A(\theta_0) = \frac{\left(\frac{d\mu}{d\theta}\right)^2}{V(A)} = \frac{\left(\left.\frac{\partial^2 \lambda_{s,\theta}}{\partial s \partial \theta}\right|_{s=0,\theta=\theta_0}\right)^2}{\left.\frac{\partial^2 \lambda_{s,\theta}}{\partial^2 s^2}\right|_{s=0}}$$

Towards a general quantum CLT for quantum Markov chains

- Both quantum and classical Fisher informations rely on an underlying CL behavior

$$I^A = \frac{\left(\frac{d\mu}{d\theta}\right)^2}{\mathrm{Var}_V(A)} \quad \leq \quad F = 4\mathrm{Var}_V(G)$$

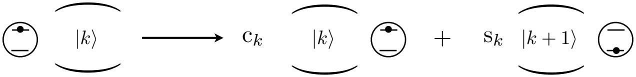
- Open questions
  - Which measurement achieves the quantum Fisher information?
  - Is there an "invariance principle" for quantum Markov chains ?

Conjecture: general CLT

Observable $X \in M(\mathbb{C}^D) \otimes M(\mathbb{C}^k)^{\otimes r}$ "localised" in system and noise units.
There is a CCR algebra with canonical coordinates $B(X)$ and Gaussian state Φ such that

$$\begin{aligned} \Phi(B(X)B(Y)) &= \lim_{n\to\infty} \langle \mathbb{F}_n(X) | \mathbb{F}_n(Y) \rangle \\ [B(X), B(Y)] &= \lim_{n\to\infty} \langle [\mathbb{F}_n(X), \mathbb{F}_n(Y)] \rangle \mathbf{1} \end{aligned}$$

The atom maser[23]



- Jaynes-Cummings interaction between a two-level atom and a cavity

$$U : |1\rangle \otimes |k\rangle \mapsto \cos\left(\phi\sqrt{k+1}\right) |1\rangle \otimes |k\rangle + \sin\left(\phi\sqrt{k+1}\right) |0\rangle \otimes |k+1\rangle$$

- Coarse grained cavity dynamics for Poisson distributed input atoms with rate N_{ex}

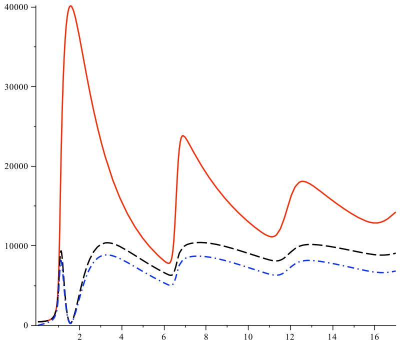
$$\frac{d\rho}{dt} = \mathcal{L}(\rho) = \sum_{i=1}^{4} \left(L_i \rho L_i^* - \frac{1}{2}\{L_i^* L_i, \rho\} \right)$$

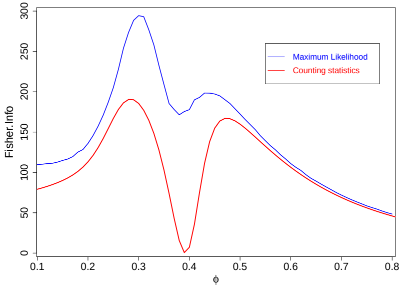
with jump operators

  - $L_1 : |k\rangle \mapsto \sqrt{N_{ex}} \sin(\phi\sqrt{k+1})\, |k+1\rangle$ (excitation absorbed from atom)
  - $L_2 : |k\rangle \mapsto \sqrt{N_{ex}} \cos(\phi\sqrt{k+1})\, |k\rangle$ (atom remains in excited state)
  - $L_3 : |k\rangle \mapsto \sqrt{k(\nu+1)}\, |k-1\rangle$ (photon emitted in the bath)
  - $L_4 : |k\rangle \mapsto \sqrt{(k+1)\nu}\, |k+1\rangle$ (photon absorbed from the bath)

[23] H.-J. Briegel, B.-G. Englert, N. Sterpi, and H. Walther, Phys. Rev. A 1994

The many Fisher informations of the atom maser [24,25]





red: Quantum Fisher info

black: observe cavity+bath

blue:observe cavity

red: Fisher info total counts

blue: Fisher info counting process

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[24] C. Catana, M van Horssen, M.G., *Phil. Trans. Royal Soc. A* (2012)

[25] C.Catana, T. Kypraios and M.G. arXiv:1311.4091

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Heisenberg scaling for systems with multiple “dynamical phases”

- Simplest example of Markov dynamics with degenerate stationary states ("phases")

$$V: \quad \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$$

$$V: \quad |0\rangle \mapsto \quad |0\rangle \otimes |0\rangle$$
$$V: \quad |1\rangle \mapsto e^{i\theta}|1\rangle \otimes |1\rangle$$

- Output + system state exhibits Heisenberg scaling for initial state $|0\rangle + |1\rangle/\sqrt{2}$

$$|\Psi^n(\theta)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle^{\otimes n} + e^{in\theta}|1\rangle \otimes |1\rangle^{\otimes n}\right)$$

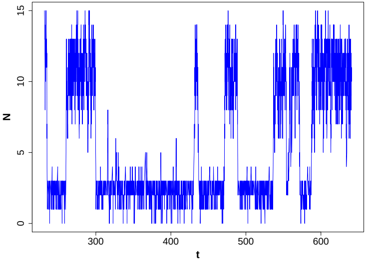
Heisenberg scaling for Markov systems with degenerate stationary phases

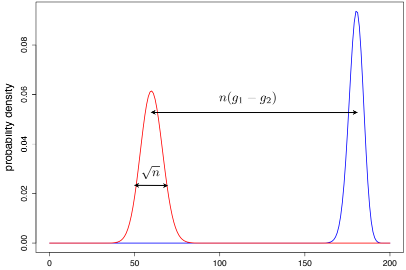
- If system has 2 stationary "phases" then O+S state is a "massive superposition"

$$|\Psi_\theta(n)\rangle = \frac{1}{\sqrt{2}}\left(|\Psi_\theta^{(0)}(n)\rangle + |\Psi_\theta^{(1)}(n)\rangle\right)$$

- Fisher information $= \mathrm{Var}(G)$ w.r.t. a mixture of Gaussian distributions

$$F \approx n^2(g_1 - g_2)^2$$





- Exploit this for quantum enhanced metrology with open systems near dynamical phase transitions [26] [27]

[26] C. Catana and M.G., *Phys. Rev. A* 2014

[27] K. Macieszczak, J.P. Garrahan, I. Lesanovsky and M.G., in preparation

Outlook

- Stationary (primitive) quantum Markov chains can be characterised completely up to unitary "change of coordinates" by measuring the output
- The output state is asymptotically Gaussian with quantum Fisher information equal to the "Markov variance of the generator"
- Multiple phases chains can exhibit Heisenberg scaling
- Future work
  - enhanced metrology & dynamical phase transitions
  - general quantum Markov CLT
  - use of feedback in system identification
  - design better input states